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# On 2-local isometries on continuous vector-valued function spaces

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ABSTRACT

A (not necessarily linear) mapping $\Phi$ from a Banach space $X$ to a Banach space $Y$ is said to be a 2-*local isometry* if for any pair $x, y$ of elements of $X$, there is a surjective linear isometry $T : X \to Y$ such that $Tx = \Phi x$ and $Ty = \Phi y$. We show that under certain conditions on locally compact Hausdorff spaces $Q$, $K$ and a Banach space $E$, every 2-local isometry on $C_0(Q, E)$ to $C_0(K, E)$ is linear and surjective. We also show that every 2-local isometry on $\ell^p$ is linear and surjective for $1 \leqslant p < \infty$, $p \neq 2$, but this fails for the Hilbert space $\ell^2$.



A (not necessarily linear) mapping $\Phi$ from a Banach space $X$ to a Banach space $Y$ is said to be a 2-*local isometry* if for any pair $x, y$ of elements of $X$, there is a surjective linear isometry $T : X \to Y$ such that $\Phi x = Tx$ and $\Phi y = Ty$. The general question is whether $\Phi$ must itself be a surjective linear isometry. This type of problem is basic in that it asks whether a local assumption is enough to guarantee a more global conclusion. Early investigations along these lines involved derivations and automorphisms of operator algebras and were carried out by Kadison [8], Larson [9], and Larson and Sourour [10]. A set $\mathcal{S}$ of operators is called *algebraically reflexive* if $\mathcal{S}$ must contain every $T$ which is local in this sense: given $x$ in the domain, there is an $S \in \mathcal{S}$ such that $Tx = Sx$. If the group $\mathcal{G}(X)$ of surjective linear isometries on $X$ is algebraically reflexive, we will say that $X$ is *iso-reflexive*. This language could also be applied to a pair $(X, Y)$ of Banach spaces if the isometries go from $X$ to $Y$.

Results concerning iso-reflexivity of certain operator algebras and function algebras have been obtained, about which [2,6,13], with their references, serve as a good introduction. In particular, Molnár and Zalar [12] showed that if $Q$ is compact, Hausdorff, and first countable, then $C(Q)$ is iso-reflexive. Jarosz and Rao [6] extended this to the vector-valued case, proving that if $Q$ is a first countable compact Hausdorff space and $E$ is a uniformly convex and iso-reflexive Banach space, then $C(Q, E)$ is iso-reflexive.

The notion of 2-local is due to Šemrl [14] who was interested in dropping the linearity assumption for local automorphisms and derivations on $\mathcal{L}(H)$, the bounded linear operators on $H$, where $H$ is a Hilbert space. To compensate for the loss of linearity, it was useful to require the local condition at two points. Molnár [11] showed that every 2-local isometry on $\mathcal{L}(H)$ is linear and so a surjective linear isometry. Gyory [5] showed that if $Q$ is a first countable, $\sigma$-compact, (separable) locally compact Hausdorff space, then every 2-local isometry on $C_0(Q)$ is a surjective linear isometry. That paper is the inspiration for the current note, in which we wish to consider the extension of Gyory's theorem to $C_0(Q, E)$ for an appropriate Banach space $E$. By $C_0(Q, E)$ we mean, of course, the continuous functions on $Q$ to $E$ which vanish at infinity and given the sup norm. In case $E$ is the scalar field, we just write $C_0(Q)$.

Let us agree to say that a Banach space $X$ is 2-*iso-reflexive* if every 2-local isometry on $X$ is both linear and surjective. We begin by considering $\ell^p$ spaces.

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**Theorem 1.** *The Banach space $\ell^p$ is 2-iso-reflexive for $1 \leqslant p < \infty$, $p \neq 2$.*

**Proof.** Suppose $\Phi$ is a 2-local isometry on $\ell^p$. First we note that $\Phi$ must be homogeneous. (In fact, this is true for a 2-local isometry on any Banach space.) For, if $x$ is given and $\lambda$ is a scalar, then there is a surjective linear isometry $T$ for which $Tx = \Phi x$ and $T(\lambda x) = \Phi(\lambda x)$. Hence,

$$\Phi(\lambda x) = T(\lambda x) = \lambda T x = \lambda \Phi x.$$

We recall that every surjective linear isometry $T$ on $\ell^p$ is what has been called a *permutation isometry*; i.e., the $j$th coordinate of $Tx$ is given by

$$Tx(j) = \lambda_j x\big(\pi(j)\big),$$

where $\pi$ is a permutation of the positive integers, and $\lambda_j$ is a modulus one scalar for each $j$. Such an operator is a unitary operator on $\ell^2$.

Suppose now that $x, y$ are elements of $\ell^p$ with finite support. Let $T$ be a surjective linear operator on $\ell^p$ such that $\Phi x = Tx$ and $\Phi y = Ty$. Thus $\Phi x$ and $\Phi y$ also have finite support and we may think of them as elements of $\ell^2$ to which the ordinary inner product $\langle\cdot,\cdot\rangle$ may be applied. Therefore, we have

$$\langle \Phi x, \Phi y\rangle = \langle Tx, Ty\rangle = \langle x, T^*Ty\rangle = \langle x, y\rangle.$$

This holds for any pair of elements with finite support. Using the linearity in both arguments of the inner product, we conclude that

$$\big\langle \Phi(x+y) - \Phi x - \Phi y, \Phi(x+y) - \Phi x - \Phi y\big\rangle = 0$$

from which it follows that

$$\Phi(x+y) - \Phi x - \Phi y = 0.$$

Hence $\Phi$ is additive on elements of finite support, and since a 2-local isometry is necessarily continuous, the density of the elements of finite support yields the fact that $\Phi$ is additive, and hence linear on $\ell^p$. The proof of the theorem is complete once we observe that a 2-local isometry is also a local isometry, and every linear local isometry on $\ell^p$, $p \neq 2$, is surjective [13, Theorem 2.1]. $\square$

**Remark 2.**

(i) The argument above was inspired by the proof given by Molnár [11] for 2-local isometries on $\mathcal{L}(H)$.

(ii) The form of linear, surjective isometries on $\ell^p$ is almost folklore. Specific references may be found on pages 96–97 and 130–131 in [4]. In fact, the arguments concerning linearity in the above proof would work on the more general sequence spaces whose coordinate vectors form a one-unconditional basis, called *admissable* spaces in [4]. In particular, $c_0$ would be included. The $p = 2$ case must only be excluded in showing the surjectivity.

(iii) In fact, infinite dimensional Hilbert spaces are not 2-iso-reflexive. Given any two pairs $x, q$ and $z, v$ of orthogonal norm-one vectors, there is a unitary $U$ such that $Ux = z$ and $Uq = v$. Let $S$ be a nonsurjective linear isometry (for example, a unilateral shift). Given $x$, $y$ of norm 1, let $q = (y - \langle y, x\rangle x)/\|y - \langle y, x\rangle x\|$. Then $Sx$, $Sq$ are orthonormal, and there exists a unitary $U$ such that $Ux = Sx$ and $Uq = Sq$. From this it is easily seen that $Uy = Sy$, and $S$ is 2-local.

As we mentioned earlier, Gyory [5] has shown that under the right conditions on $Q$, the function space $C_0(Q)$ is 2-iso-reflexive. Using his ideas, we now extend the result to the vector-valued case. It is natural to ask whether for $C_0(Q, E)$ to be 2-iso-reflexive it is necessary for $E$ to be 2-iso-reflexive. The positive answer is not difficult to verify.

**Theorem 3.** *If $C_0(Q, E)$ is 2-iso-reflexive, then $E$ is 2-iso-reflexive.*

**Proof.** Suppose $E$ is not 2-iso-reflexive, in which case there exists an operator $V$ on $E$ which is 2-local but not both linear and surjective. Define $\Phi$ on $C_0(Q, E)$ to itself by

$$\Phi F(t) = V\big(F(t)\big).$$

Let $F, G$ be given elements of $C_0(Q, E)$. For each $t \in Q$, there exists a linear, surjective isometry $V(t)$ on $E$ for which

$$V(t)F(t) = VF(t) \quad \text{and} \quad V(t)G(t) = VG(t).$$

The operator $T$ defined on $C_0(Q, E)$ by $TH(t) = V(t)H(t)$ for $H \in C_0(Q, E)$ is linear and surjective. Furthermore,

$$\Phi F(t) = TF(t) \quad \text{and} \quad \Phi G(t) = TG(t).$$

Hence, $\Phi$ is a 2-local isometry, but it cannot be both linear and surjective, and we infer that $C_0(Q, E)$ is not 2-iso-reflexive. $\Box$

We are ready now to tackle the main question about the 2-iso-reflexivity of $C_0(Q, E)$. We will make use of the arguments of Gyory [5] in several places. The strategy is to try to describe the form of a 2-local isometry and in this way show it is actually linear and surjective. Such descriptions usually involve the construction of a map on the topological space $Q$ to itself which often turns out to be a homeomorphism. We are going to consider the case where the 2-local isometry $\Phi$ is assumed to go from $C_0(Q, E)$ to $C_0(K, E)$, where $Q$ and $K$ are locally compact Hausdorff spaces with certain properties. There are several methods that have been used to construct the hoped-for homeomorphism, and we will follow Gyory's approach in this situation. The vector-valued case presents some difficulties and to avoid them we will also make use of the isometry $\Psi$ defined from $C_0(Q, E)$ onto a subspace $\mathcal{M}$ of $C_0(Q \times B(E^*))$ by

$$\Psi F(s, v^*) = v^*\big(F(s)\big) \quad \text{for } F \in C_0(Q, E),$$

where the unit ball $B(E^*)$ is endowed with the weak*-topology [7]. It is not difficult to show that the Choquet Boundary, $\mathrm{ch}(\mathcal{M})$, of $\mathcal{M}$ as a subspace of $C_0(Q \times B(E^*))$ is the set of all pairs $(s, x^*)$, where $s \in Q$ and $x^*$ is an extreme point of the unit ball of $E^*$, denoted by $\mathrm{ext}(E^*)$. Recall that the Choquet Boundary of a subspace $Y$ of a $C_0(Q)$ space is the set of all $s \in Q$ such that the evaluation functional $\psi_s$ is an extreme point of the unit ball of $Y^*$, denoted by $\mathrm{ext}(Y^*)$. The extreme points of a subspace of a vector-valued space $C_0(Q, E)$ are known to be of the form $x^* \circ \psi_s$, where $s \in Q$ and $x^* \in \mathrm{ext}(E^*)$. (See [3, Chapter 2] and its references.)

If $\Phi$ is a 2-local isometry from $C_0(Q, E)$ to $C_0(K, E)$, then we can define a 2-local isometry $\Phi_0$ from the subspace $\mathcal{M}$ to the subspace $\mathcal{N} = \Psi(C_0(K, E))$ by

$$\Phi_0 f(t, v^*) = \Psi \Phi \Psi^{-1} f(t, v^*). \tag{1}$$

For $(s, x^*) \in \mathrm{ch}(\mathcal{M})$ and $f \in \mathcal{M}$, we define

$$\mathcal{A}_{s,x^*,f} = \big\{(t, v^*, \lambda):\ \Phi_0 f(t, v^*) = \lambda f(s, x^*),\ |\lambda| = 1\big\}.$$

**Lemma 4.** *Let $Q$, $K$ be locally compact Hausdorff spaces and suppose $E$ is a smooth, reflexive Banach space. If $\Phi$ is a 2-local isometry from $C_0(Q, E)$ to $C_0(K, E)$ with $\Psi, \Phi_0, \mathcal{M}, \mathcal{N}$ as defined above, there exist a subset $\mathcal{K}_0$ of $\mathrm{ch}(\mathcal{N})$, a subset $K_0$ of $K$, a modulus one-valued function $h$ on $\mathcal{K}_0$, and a function $\varphi_1$ from $K_0$ onto $Q$ such that for each $(t, w^*) \in \mathcal{K}_0$, there is a pair $(s, x^*) \in \mathrm{ch}(M)$ satisfying the equality*

$$w^*\big[\Phi F(t)\big] = h(t, w^*) x^*\big[F\big(\varphi_1(t)\big)\big], \quad \textit{for all } F \in C_0(Q, E),$$

*where $\varphi_1(t) = s$.*

**Proof.** First we will need the description of a surjective linear isometry from the subspace $\mathcal{M}$ to the subspace $\mathcal{N}$. Suppose $T$ is such an isometry. Then $T^*$ is an isometry from $\mathcal{N}^*$ onto $\mathcal{M}^*$ which must take extreme points of the unit ball of $\mathcal{N}^*$ to extreme points of the unit ball of $\mathcal{M}^*$. Given $(t, w^*)$ in $\mathrm{ch}(\mathcal{N})$, we have

$$T^*\big(\psi_{(t,w^*)}\big) = \lambda \psi_{(s,x^*)} \quad \text{where } |\lambda| = 1 \text{ and } (s, x^*) \in \mathrm{ch}(\mathcal{M}). \tag{2}$$

If we suppose that $\psi_{(s,x^*)} = \psi_{(r,y^*)}$, with $r \neq s$, then we can find $F \in \mathcal{M}$ so that $x^*(F(s)) \neq 0$ but $y^*(F(r)) = 0$, so that it is necessary that $r = s$. However, it is true that $\psi_{(s,e^{i\theta}x^*)} = e^{i\theta}\psi_{(s,x^*)}$. To get around this difficulty, for each $x^* \in \mathrm{ext}(E^*)$, let $\Gamma(x^*) = \{e^{i\theta}x^*:\ 0 \leqslant \theta < 2\pi\}$. The collection $\{\Gamma(x^*):\ x^* \in \mathrm{ext}(E^*)\}$ is a set of equivalence classes and we let $\tau$ be a selection function so that $\tau(x^*) \in \Gamma(x^*)$. If in Eq. (2) we replace $w^*$ by $\tau(w^*)$ and suppose the corresponding $x^* = e^{i\alpha}\tau(x^*)$, we have

$$T^*\big(\psi_{(t,\tau(w^*))}\big) = \lambda e^{i\alpha}\psi_{(s,\tau(x^*))}. \tag{3}$$

Define $\varphi(t, \tau(w^*)) = (s, \tau(x^*))$ from the above equation and more generally, let

$$\varphi(t, w^*) = \big(s, e^{i\theta}\tau(x^*)\big) \quad \text{if } w^* = e^{i\theta}\tau(w^*).$$

This gets $\varphi$ well defined on $\mathrm{ch}(\mathcal{N})$ to $\mathrm{ch}(\mathcal{M})$ and it must be onto $\mathrm{ch}(\mathcal{M})$ because $(T^*)^{-1}$ is also an isometry. If we define $h(t, w^*) = \lambda e^{i\alpha}$ from (3), then it can be seen that for each $(t, w^*) \in \mathrm{ch}(\mathcal{N})$

$$Tf(t, w^*) = h(t, w^*) f\big(\varphi(t, w^*)\big) \quad \text{for all } f = \Psi F \in \mathcal{M}. \tag{4}$$

Given $(s, x^*) \in \mathrm{ch}(\mathcal{M})$ and $f \in \mathcal{M}$, by the 2-locality of $\Phi_0$, there is a linear isometry $T$ from $\mathcal{M}$ onto $\mathcal{N}$ of the form described in (4), with $Tf = \Phi_0 f$, so we may select $(t, w^*) \in \mathrm{ch}(\mathcal{N})$ such that $\varphi(t, w^*) = (s, x^*)$. Thus $(t, w^*, \lambda) \in \mathcal{A}_{s,x^*,f}$, where $\lambda$ is the $h(t, w^*)$ guaranteed by (4).

The sets $\mathcal{A}_{s,x^*,f}$ are therefore closed, nonempty, and for a function $f$ such that $f(s,x^*)=1$, $\mathcal{A}_{s,x^*,f}$ is contained in a compact subset of $K \times B(E^*) \times \mathbb{C}$. It will then follow that if we can show that the collection has the finite intersection property, then the intersection

$$\mathcal{A}_{s,x^*} = \bigcap_{f \in \mathcal{M}} \mathcal{A}_{s,x^*,f}$$

will be nonempty. To that end, let $(s,x^*) \in \mathrm{ch}(\mathcal{M})$ and let $f_1 = \Psi F_1, \ldots, f_n = \Psi F_n$ be elements of $\mathcal{M}$. We define a function $g \in C_0(Q)$ by

$$g(\cdot) = \big(2\|F_1\| - \big\|F_1(\cdot) - F_1(s)\big\|\big) + \cdots + \big(2\|F_n\| - \big\|F_n(\cdot) - F_n(s)\big\|\big) + 1. \tag{5}$$

We observe that $g(r) \geqslant 1$ for all $r$ and that $g$ assumes its maximum at $r = s$. Furthermore, if $g$ assumes a max at $q$, then we have $F_j(q) = F_j(s)$ for $j = 1, 2, \ldots, n$. Now let $u \in E$ be such that $\|u\| = 1$ with $x^*(u) = 1$ and choose a function $f_0 \in C_0(Q)$ such that $f_0 : Q \to [0,1]$ with $f_0(s) = 1$. Finally, we define $F_0 = f_0 \otimes u$ (which means, as usual that $(f_0 \otimes u)(s) = f_0(s)u$), $F = gF_0$ and $f = \Psi F$. Suppose $(t, w^*, \lambda) \in \mathcal{A}_{s,x^*,f}$ so that we have

$$\Phi_0 f(t,w^*) = \lambda f(s,x^*) = \lambda g(s). \tag{6}$$

Let $j \in \{1,\ldots,n\}$ be fixed. Since $\Phi_0 = \Psi \Phi \Psi^{-1}$ is 2-local, there exists a linear surjective isometry $T_{f,f_j}$ from $\mathcal{M}$ onto $\mathcal{N}$ such that $T_{f,f_j} f = \Phi_0 f$ and $T_{f,f_j} f_j = \Phi_0 f_j$. Our objective is to show that $(t,w^*,\lambda) \in \mathcal{A}_{s,x^*,f_j}$. Using (6) and (4), we know that there are functions $h_{fj}$ and $\varphi_{fj}$ defined on $\mathrm{ch}(\mathcal{N})$ such that

$$\lambda g(s) = \Phi_0 f(t,w^*) = T_{f,f_j} f(t,w^*) = h_{fj}(t,w^*) f\big(\varphi_{fj}(t,w^*)\big). \tag{7}$$

If we suppose that $\varphi_{fj}(t,w^*) = (r,y^*)$, and apply absolute values to both sides of the above equations, we obtain

$$g(s) = \big|\Psi F(r,y^*)\big| = \big|y^*\big(F(r)\big)\big| = \big|g(r) f_0(r) y^*(u)\big| \leqslant g(s).$$

It follows from this equality that $g(r) = g(s)$, $f_0(r) = 1$, and $y^*(u) = e^{i\theta}$ for some $\theta$ between 0 and $2\pi$. As we noted above, $g(r) = g(s)$ implies that $F_j(r) = F_j(s)$. Also, we see that

$$x^*(u) = e^{-i\theta} y^*(u) = 1$$

and by the smoothness of $E$, we have $x^* = e^{-i\theta} y^*$. From (7) we get the equation

$$\lambda g(s) = h_{fj}(t,w^*) f(r,y^*) = h_{fj}(t,w^*) g(r) y^*(u).$$

Since $g(s) = g(r) \neq 0$, we conclude that

$$\lambda = h_{fj}(t,w^*) y^*(u). \tag{8}$$

Now,

$$\begin{aligned} \Phi_0 f_j(t,w^*) &= h_{fj}(t,w^*) f_j\big(\varphi_{fj}(t,w^*)\big) = h_{fj}(t,w^*) f_j(r,y^*) \\ &= h_{fj}(t,w^*) y^*\big(F_j(r)\big) \\ &= h_{fj}(t,w^*) y^*(u) x^*\big(F_j(s)\big) \\ &= \lambda f_j(s,x^*), \end{aligned}$$

where we have utilized (8) and the fact that $F_j(r) = F_j(s)$. This says that $(t,w^*,\lambda) \in \mathcal{A}_{s,x^*,f_j}$ and since $j$ was arbitrary in $\{1,2,\ldots,n\}$, the f.i.p. is satisfied, and $\mathcal{A}_{s,x^*} \neq \emptyset$.

Again borrowing the notation of Gyory [5], for $(s,x^*) \in \mathrm{ch}(\mathcal{M})$, we let

$$B_{s,x^*} = \big\{(t,w^*) \in \mathrm{ch}(\mathcal{N}){:}\ (t,w^*,\lambda) \in \mathcal{A}_{s,x^*} \text{ for some } \lambda\big\},$$

and

$$\mathcal{K}_0 = \big\{(t,w^*) \in \mathrm{ch}(\mathcal{N}){:}\ (t,w^*) \in B_{s,x^*} \text{ for some } (s,x^*) \in \mathrm{ch}(\mathcal{M})\big\}.$$

Suppose $(t,w^*) \in B_{s,x^*}$ and $(t,v^*) \in B_{r,y^*}$, where we assume that $r \neq s$. This means that

$$w^*\big[\Phi F(t)\big] = \lambda_1 x^*\big(F(s)\big) \quad \text{for all } F \in C_0(Q,E), \tag{9}$$

and

$$v^*\big[\Phi F(t)\big] = \lambda_2 y^*\big(F(r)\big) \quad \text{for all } F \in C_0(Q,E), \tag{10}$$

where $|\lambda_1| = |\lambda_2| = 1$. Since $E$ is reflexive, we may choose $u, v \in E$, each with norm one, such that

$$\lambda_1 x^*(u) = 1 \quad \text{and} \quad \lambda_2 y^*(v) = 1.$$

Now choose $F$ in $C_0(Q, E)$ such that $\|F\| = 1$ with $F(s) = u$ and $F(r) = v$. From (9), (10), and the choice of $F$ we have

$$w^*\big[\Phi F(t)\big] = 1 = v^*\big[\Phi F(t)\big].$$

Since $\Phi$ is an isometry, we must have $\|\Phi F(t)\| = 1$, so that by the smoothness of $E$, we conclude that $w^* = v^*$. But if we now choose an $F$ with $F(s) = u$ where $x^*(u) \neq 0$ and $F(r) = 0$, then (9) and (10) cannot both hold. We are forced to conclude that $r = s$. The happy conclusion of this is that given $(t, w^*) \in \mathcal{K}_0$, there is a unique $s \in Q$ such that $(t, w^*) \in B_{s,x^*}$ for some $x^* \in \text{ext}(E^*)$. Let $K_0$ be the set of all $t \in K$ such that $(t, w^*) \in \mathcal{K}_0$ for some $w^* \in \text{ext}(E^*)$. We define a function $\varphi_1$ from $K_0$ to $Q$ by

$$\varphi_1(t) = s,$$

where the $s$ is as described above. Then $\varphi_1$ is onto $Q$ and for every $(t, w^*) \in \mathcal{K}_0$ there is $x^* \in \text{ext}(E^*)$ and $\lambda$ with $|\lambda| = 1$ such that

$$w^*\big[\Phi F(t)\big] = \lambda x^*\big(F\big(\varphi_1(t)\big)\big) \quad \text{for all } F \in C_0(Q, E). \tag{11}$$

Let $h(t, w^*) = \lambda$ from the equation above. Note that if $v^*$ is a multiple of $w^*$ above, then $(t, v^*) \in \mathcal{K}_0$ and

$$v^*\big[\Phi F(t)\big] = \lambda e^{i\theta} x^*\big(F\big(\varphi_1(t)\big)\big)$$

for some $\theta$ and all $F$, so that we may take $h(t, v^*) = h(t, w^*)$. Hence we have, finally, that for any $(t, w^*) \in \mathcal{K}_0$, there is a pair $(s, x^*) \in \text{ch}(\mathcal{M})$, where $s = \varphi_1(t)$ such that

$$w^*\big[\Phi F(t)\big] = h(t, w^*) x^*\big[F\big(\varphi_1(t)\big)\big] \quad \text{for all } F \in C_0(Q, E). \qquad \square \tag{12}$$

Our next goal is to obtain a vector-valued version of Theorem 1 in [5]. To do so, we will assume that the space $Q$ is first countable, and that the space $E$ is smooth, reflexive, and 2-iso-reflexive. The first countable condition is required in order to be able to construct functions that attain their norm at a single point. We recall the characterization of linear surjective isometries on $C_0(Q, E)$ due to Behrends [1, Theorem 8.10]. (See also Chapter 7 in [4].)

**Theorem 5** *(Behrends). Let $T$ be a linear isometry from $C_0(Q, X)$ onto $C_0(K, Y)$ where $Q, K$ are locally compact Hausdorff spaces and $X, Y$ are Banach spaces with trivial centralizers. Then there exists a homeomorphism $\varphi$ from $K$ onto $Q$ and a continuous function $t \to V(t)$ from $K$ to the group of isometries from $X$ onto $Y$ given the S.O.T. such that*

$$TF(t) = V(t)F\big(\varphi(t)\big) \tag{13}$$

*for all $t \in K$ and $F \in C_0(Q, X)$.*

We will not discuss centralizers here. It is enough to know that a smooth space has a trivial centralizer [1, Proposition 5.1].

**Theorem 6.** *Let $Q, K$ be locally compact Hausdorff spaces with $Q$ first countable, let $E$ be a smooth, reflexive Banach space which is 2-iso-reflexive, and suppose $\Phi$ is a 2-local isometry from $C_0(Q, E)$ into $C_0(K, E)$. Then there exists a subset $K_0$ of $K$, a continuous bijection $\varphi$ from $K_0$ onto $Q$, and a mapping $t \to V(t)$, which is continuous from $K_0$ into the space $\mathcal{G}(E)$ of surjective linear isometries on $E$ with the strong operator topology (S.O.T.), such that*

$$\Phi F(t) = V(t)F\big(\varphi(t)\big) \quad \textit{for all } t \in K_0 \textit{ and } F \in C_0(Q, E).$$

**Proof.** We let $K_0$ be the same as the set $K_0$ in Lemma 4 and $\varphi$ will be the function called $\varphi_1$ in the lemma. We already know that $\varphi$ is surjective and so let us suppose that $\varphi(t) = \varphi(r) = s \in Q$, where $t, r \in K_0$. Then by Lemma 4, there exist $w^*, v^*, x^*, y^* \in \text{ext}(E^*)$ such that

$$w^*\big[\Phi F(t)\big] = h(t, w^*) x^*\big(F\big(\varphi(t)\big)\big) \tag{14}$$

and

$$v^*\big[\Phi F(r)\big] = h(r, v^*) y^*\big(F\big(\varphi(r)\big)\big) \tag{15}$$

for all $F \in C_0(Q, E)$. Choose $u, v$ in the unit sphere of $E$ such that $x^*(u) = 1 = y^*(v)$ and let $f \in C_0(Q)$ be a function from $Q$ into $[0, 1]$ for which $f(s) = 1$ and $f$ peaks only at $s$, that is, $f(q) < 1$ if $q \neq s$. If $F = f \otimes u$ and $G = f \otimes v$, then $F$ and $G$ attain their norms of 1 only at $s$. It now follows from (14) and (15) that

$$\big|w^*\big[\Phi F(t)\big]\big| = 1 = \big|v^*\big[\Phi G(r)\big]\big|. \tag{16}$$

Since $\Phi$ is 2-local, it agrees with a surjective linear isometry at both $F$ and $G$. By Behrends' theorem we have for each $t \in K$ a surjective linear isometry $V_{F,G}(t)$ of $E$ and a homeomorphism $\varphi_{F,G}$ of $K$ onto $Q$ such that

$$\Phi F(t) = V_{F,G}(t)F\big(\varphi_{F,G}(t)\big)$$

and

$$\Phi G(r) = V_{F,G}(r)G\big(\varphi_{F,G}(r)\big).$$

From (16), we see that $\|\Phi F(t)\| = 1 = \|\Phi G(r)\|$, and by the equations just above, and the fact that $V_{F,G}(t)$ is an isometry, we must have

$$1 = \big\|\Phi F(t)\big\| = \big\|F\big(\varphi_{F,G}(t)\big)\big\|$$

so that $\varphi_{F,G}(t) = s$ since $F$ attains its norm only at $s$. Similarly, we have $\varphi_{F,G}(r) = s$ and since $\varphi_{F,G}$ is a homeomorphism, we conclude that $t = r$ and $\varphi$ is a bijection.

Now for each $t \in K_0$ define an operator $V(t)$ on $E$ by

$$V(t)u = \Phi F(t) \tag{17}$$

if $u \in E$ and $F(\varphi(t)) = u$. We must show that $V(t)$ is well defined. Let $F$ be any element of $C_0(Q, E)$ for which $F(\varphi(t)) = u$ and choose $G \in C_0(Q, E)$ such that $G(\varphi(t)) = u$, $\|G\| = \|u\|$ and $G$ peaks only at $\varphi(t)$. As above, there must exist a homeomorphism $\varphi_{F,G}$ from $K$ onto $Q$ and a map $r \to V_{F,G}(r)$ from $K$ into $\mathcal{G}(E)$ such that

$$\Phi F(t) = V_{F,G}(t)\big[F\big(\varphi_{F,G}(t)\big)\big] \quad \text{and} \quad \Phi G(t) = V_{F,G}(t)\big[G\big(\varphi_{F,G}(t)\big)\big]. \tag{18}$$

We also have from Lemma 4 that if $x^*(u) = \|u\|$, there is some $w^* \in \operatorname{ext}(E^*)$ such that

$$w^*\big[\Phi G(t)\big] = h(t, w^*)x^*\big[G\big(\varphi(t)\big)\big]. \tag{19}$$

Taking absolute values of both sides in (19) we obtain $|w^*[\Phi G(t)]| = \|u\|$, and if we use the other form for $\Phi G(t)$ from (18) and the fact that $V_{F,G}(t)$ is an isometry, we get

$$\|u\| = \big|w^*\big[\Phi G(t)\big]\big| = \big|w^*\big[V_{F,G}(t)\big(G\big(\varphi_{F,G}(t)\big)\big)\big]\big| \leqslant \big\|G\big(\varphi_{F,G}(t)\big)\big\| \leqslant \|u\|.$$

Since $G$ peaks only at $\varphi(t)$, we must have $\varphi_{F,G}(t) = \varphi(t)$, from which it follows that

$$\Phi F(t) = V_{F,G}(t)F\big(\varphi(t)\big) = V_{F,G}(t)G\big(\varphi(t)\big) = \Phi G(t).$$

This proves that $V(t)$ is well defined and the equation

$$\Phi F(t) = V(t)F\big(\varphi(t)\big) \tag{20}$$

automatically results from the definition of $V(t)$. Now let $u, v \in E$ and $t \in K_0$. Let $F, G$ be elements of $C_0(Q, E)$ which peak only at $\varphi(t)$ and suppose $F(\varphi(t)) = u$ and $G(\varphi(t)) = v$. Then Eq. (18) holds for this $F$ and $G$. As in our earlier argument, $\varphi_{F,G}(t) = \varphi(t)$ so that

$$V(t)u = \Phi F(t) = V_{F,G}(t)u \quad \text{and} \quad V(t)v = \Phi G(t) = V_{F,G}(t)v,$$

proving that $V(t)$ is 2-local. Now because $E$ is 2-iso-reflexive, we conclude that $V(t)$ is a surjective linear isometry on $E$ and this is true for each $t \in K_0$. Hence, by (20) we have established the formula given in the statement of the theorem. To complete the proof, we must prove the continuity assertions.

The arguments we give are standard ones (see, for example [4, Chapter 7]), but let us indicate how they go. Suppose $\{t_\alpha\}$ is a net in $K_0$ converging to $t \in K_0$ for which the corresponding net $\{\varphi(t_\alpha)\}$ does not converge to $\varphi(t)$. Then there is an open neighborhood $U$ of $\varphi(t)$ and a subnet $\{t_{\alpha_\beta}\}$ for which $\varphi(t_{\alpha_\beta}) \in Q \setminus U$ for all $\beta$. Let $u \in E$ be nonzero, and let $F$ be an element of $C_0(Q, E)$ such that $F(\varphi(t)) = u$ and $F(r) = 0$ for $r \in Q \setminus U$. Now $\Phi F$ is continuous, so $\Phi F(t_{\alpha_\beta}) \to \Phi F(t)$. However, for all $\beta$, $\Phi F(t_{\alpha_\beta}) = V(t_{\alpha_\beta})0 = 0$, while $\Phi F(t) = V(t)u \neq 0$. This contradiction establishes the continuity of $\varphi$.

Again suppose $t_\alpha \to t$ in $K_0$, and let $u \in E$. Given any neighborhood $W$ of $\varphi(t)$, there is a compact neighborhood $U$ of $\varphi(t)$ and a function $f \in C_0(Q)$ such that $f : Q \to [0, 1]$ with $f(r) = 1$ for all $r \in U$. If $F = f \otimes u$, we have, for $\alpha$ sufficiently far out in the net,

$$\big\|V(t_\alpha)u - V(t)u\big\| = \big\|\Phi F(t_\alpha) - \Phi F(t)\big\| \to 0.$$

Thus $t \to V(t)$ is continuous from $K_0$ to $\mathcal{G}(E)$ in S.O.T. $\square$

We turn now to address the proof of the theorem advertised in the beginning. It will require a bit more hypothesis on the topological spaces $Q, K$. Recall that a locally compact space is said to be *$\sigma$-compact* if it is the union of at most countably many compact spaces. We will need a very interesting lemma stated and proved by Gyory [5].

**Lemma 7** *(Gyory). Let $Q$ be a first countable $\sigma$-compact Hausdorff space and $R$ a countable subset of $Q$ with distinct elements $r_1, r_2, \ldots$. Then there exist positive functions $f, g$ in $C_0(Q)$ from $Q$ into $(0,1]$ such that $f$ has a strict maximum at every point of $R$ and $(f,g)^{-1}(f(r_n), g(r_n)) = r_n$ for each positive integer $n$.*

**Theorem 8.** *Let $Q, K$ be $\sigma$-compact metric spaces and $E$ a reflexive, smooth, 2-iso-reflexive Banach space. If $\Phi$ is a 2-local isometry from $C_0(Q,E)$ to $C_0(K,E)$, there is a homeomorphism $\varphi$ from $K$ onto $Q$ and a continuous map $t \to V(t)$ from $K$ into $\mathcal{G}(E)$ with the S.O.T. such that*

$$\Phi F(t) = V(t)F\bigl(\varphi(t)\bigr)$$

*for all $F \in C_0(Q,E)$ and $t \in K$. Thus, $\Phi$ is a surjective linear isometry.*

**Proof.** By Theorem 6, the above holds for $t \in K_0$, where we know that $\varphi$ is a continuous bijection. Our job, then, is to show that $K_0 = K$ and that $\varphi$ is a homeomorphism. We do it by making suitable adaptations to Gyory's proof of Theorem 2 in [5].

Observe that since $\Phi$ is 2-local, there must be surjective linear operators from $C_0(Q,E)$ to $C_0(K,E)$ so that by Behrends' theorem, $Q$ and $K$ are homeomorphic. If $K$ is finite, we must have $K_0 = K$, since both have the same cardinality as $Q$. Assume then, that $K$ is infinite.

A $\sigma$-compact metric space is separable, so let $\{r_n\}$ be a countable, dense subset of distinct points of $Q$, let $t_n = \varphi^{-1}(r_n)$ for each $n$, and suppose $f, g$ are the functions guaranteed by Lemma 7. For a fixed $u \in E$ with $\|u\| = 1$, let $F = f \otimes u$ and $G = g \otimes u$. By the 2-local property there must be surjective linear isometries $V_{F,G}(\cdot)$ and a homeomorphism $\varphi_{F,G}$ from $K$ onto $Q$ such that

$$V_{F,G}(t_n)F\bigl(\varphi_{F,G}(t_n)\bigr) = \Phi F(t_n) = V(t_n)F(r_n) \tag{21}$$

and

$$V_{F,G}(t_n)G\bigl(\varphi_{F,G}(t_n)\bigr) = \Phi G(t_n) = V(t_n)G(r_n). \tag{22}$$

Note that the last term in each of the equations comes from Theorem 6 and the fact that $t_n \in K_0$. Taking the norms of both sides in Eqs. (21) and (22), we obtain

$$\begin{aligned} f\bigl(\varphi_{F,G}(t_n)\bigr) &= \bigl\|F\bigl(\varphi_{F,G}(t_n)\bigr)\bigr\| = \bigl\|V_{F,G}(t_n)F\bigl(\varphi_{F,G}(t_n)\bigr)\bigr\| = f(r_n), \\ g\bigl(\varphi_{F,G}(t_n)\bigr) &= \bigl\|G\bigl(\varphi_{F,G}(t_n)\bigr)\bigr\| = \bigl\|V_{F,G}(t_n)G\bigl(\varphi_{F,G}(t_n)\bigr)\bigr\| = g(r_n). \end{aligned}$$

By the properties of $f$ and $g$ from the lemma, we must conclude that $\varphi_{F,G}(t_n) = r_n$ for each $n$, and therefore, $\varphi_{F,G}(t_n) = \varphi(t_n)$ for each $n$. Since the homeomorphism pairs $\{t_n\}$ with the dense set $\{r_n\}$, it must be the case that $\{t_n\}$ is dense in $K$, and it follows that $K_0$ is dense in $K$.

Suppose now that $t \in K$ and $\{t_n\}$ is a sequence in the dense set $K_0$ with $t_n \to t$. Assume that $\{\varphi(t_n)\}$ has no accumulation point in $Q$. Since $Q$ is $\sigma$-compact, there exists a function $f \in C_0(Q)$, $f: Q \to [0,1]$ with $f(s) \neq 0$ for each $s \in Q$. Since the sequence $\{\varphi(t_n)\}$ has no accumulation points, it can have at most finitely many terms in any compact set, and it follows that $f(\varphi(t_n)) \to 0$. For any $u \in E$ with $\|u\| = 1$, let $F = f \otimes u$, so that $\Phi F(t_n) = V(t_n)F(\varphi(t_n)) \to 0$. Since $\Phi F(t_n) \to \Phi F(t)$, we have $\Phi F(t) = 0$. The 2-local property of $\Phi$ yields a surjective linear operator $T_{F,F}$ which agrees with $\Phi$ at $F$, so we have

$$T_{F,F}F(t) = V_{F,F}(t)F\bigl(\varphi_{F,F}(t)\bigr) = 0,$$

where $V_{F,F}(t)$ is an isometry and $\varphi_{F,F}$ is a homeomorphism of $K$ onto $Q$. Hence, $0 = F(\varphi_{F,F}(t)) = f(\varphi_{F,F}(t))u$, which is impossible since $f$ is never zero. This contradiction says that there must be a subsequence (which we label as the same) $\{t_n\}$ of the original sequence and $s \in Q$ such that $\varphi(t_n) \to s$. Since the map $\varphi$ is surjective from $K_0$ to $Q$, there is some $r \in K_0$ such that $\varphi(r) = s$.

Let $F \in C_0(Q,E)$ be such that it peaks at $s$ and only $s$. Because $\Phi F$ is continuous we have $\Phi F(t_n) \to \Phi F(t)$ and also

$$\bigl\|\Phi F(t_n)\bigr\| = \bigl\|V(t_n)F\bigl(\varphi(t_n)\bigr)\bigr\| = \bigl\|F\bigl(\varphi(t_n)\bigr)\bigr\| \to \bigl\|F(s)\bigr\|,$$

so that

$$\bigl\|\Phi F(t)\bigr\| = \bigl\|F(s)\bigr\|. \tag{23}$$

Furthermore,

$$\bigl\|\Phi F(r)\bigr\| = \bigl\|V(r)F\bigl(\varphi(r)\bigr)\bigr\| = \bigl\|F(s)\bigr\|. \tag{24}$$

For this $F$ we again have a linear, surjective isometry $T_{F,F}$ as above so that

$$\Phi F(\cdot) = V_{F,F}(\cdot)F\bigl(\varphi_{F,F}(\cdot)\bigr).$$

From this and Eqs. (23) and (24) it follows that

$$\|F(\varphi_{F,F}(t))\| = \|\Phi F(t)\| = \|F(s)\| = \|\Phi F(r)\| = \|F(\varphi_{F,F}(r))\|.$$

Since $F$ peaks only at $s$, we have $\varphi_{F,F}(t) = \varphi_{F,F}(r)$ which implies that $t = r$. The conclusion is that $t \in K_0$ and $K = K_0$.

Finally, we observe that the defining equation for $\Phi$ in the statement of the theorem describes a surjective linear isometry since it holds for all $t \in K$, and the function $\varphi$ is a homeomorphism because it agrees with the homeomorphism $\varphi_{F,G}$ on a dense set. □

As a corollary of the above theorem we can obtain a partial converse of Theorem 3.

**Corollary 9.** *If $K$ is a $\sigma$-compact metric space and $E$ is a smooth reflexive Banach space, then $C_0(K, E)$ is 2-iso-reflexive if and only if $E$ is 2-iso-reflexive.*

From Theorem 1, another corollary is available.

**Corollary 10.** *If $K$ is a $\sigma$-compact metric space, then $C_0(K, \ell^p)$ is 2-iso-reflexive for $1 < p < \infty$, $p \neq 2$.*

We close with a few remarks.

**Remark 11.**

(i) We began the proof of Lemma 4 by describing the form of a linear isometry from the special subspace $\mathcal{M}$ of $C_0(Q \times B(E^*))$ onto $\mathcal{N}$. The description of isometries on subspaces of $C_0(K)$ spaces has been given and in particular such a theorem, called Novinger's theorem, was given as Theorem 2.3.10 in [3]. This theorem could have been used here except for the fact that the statement in part (i) of that theorem is not quite correct. The hypothesis that the subspace separates the points of its Choquet Boundary is not quite enough. Strong separation would do it, or if there are no $s, r$ such that $\psi_s = e^{i\theta}\psi_r$ for some fixed $\theta$. Unfortunately, the space $\mathcal{M}$ that we were using suffers exactly from that problem. We hasten to point out that part (ii) of the Theorem 2.3.10 mentioned above (and the part actually due to Novinger) is correct as stated, because in that case, the subspace is assumed to contain the constant functions. Hence, the $\theta$ involved would have to be 0, so that separation of points suffices.

(ii) It is possible to consider a different approach after the proof of Lemma 4 which does not assume that $E$ is 2-iso-reflexive. It can be shown that if $w^*[\Phi G(t)] = \|\Phi G(t)\| = \|\Phi G\|$, then $(t, w^*) \in \mathcal{K}_0$, in the notation of the lemma. As a result of this, it can be shown that if $\Phi$ is 2-local and surjective (with $Q$ and $K$ metric and $\sigma$-compact), then $\Phi$ is linear. It is enough, in fact, to assume that the span of $E(t)$ is all of $E$ for each $t \in K$, where $E(t) = \{u \in E\colon\ u = \Phi F(t) \text{ for some } F \in C_0(Q, E)\}$.

(iii) In the statement of Theorem 2 in [5], it is assumed that $Q$ is first countable and $\sigma$-compact. However, in the proof it is also assumed that the space is separable, which implies that it is metrizable.

(iv) The only need for reflexivity of $E$ in these results is to be able to know that for each extreme point $x^*$ of the unit ball of $E^*$, there is some $u \in E$ with $x^*(u) = 1$.

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